Novel Fractional Viscoelastic Model of Ligaments for High Strain Rates

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Abstract Injuries to cervical spine ligaments are a common occurrence in high-rate events. To build an accurate computational model that describes the mechanical behaviour of these ligaments at a fast strain rate, their viscoelastic behaviour needs to be accounted for. [1] provided a linear viscoelastic model of the anterior longitudinal ligament (ALL) in the cervical spine, describing the instantaneous elastic response and the relaxation behaviour using exponential reduced relaxation functions. A novel fractional order viscoelastic model is proposed, offering the potential to consistently describe viscoelastic behaviour over longer time scales while using fewer parameters. Stress and strain measurements from high-rate uniaxial bone-ligament-bone segment tensile tests were obtained from the original study, and integer and fractional order viscoelastic model results were compared with each other and the experimental stress data. The linear instantaneous elastic properties of the ligaments were confirmed, and a linear instantaneous elastic function was used in both the integer and fractional models. The fractional order model proved competitive with the traditional integer order model at short time scales. Further studies including stress relaxation at both very short and very long time scales are needed to distinguish between models.

Keywords cervical spine, fractional calculus, high-rate strain, ligament, viscoelasticity

I. INTRODUCTION

Cervical spinal ligaments play an important role in stabilising the cervical spine to prevent critical spinal injuries [2]. Injuries to these ligaments can occur in high-rate events such as accidental falls, automotive crashes, and military scenarios. A common example of ligament injury in the cervical spine is whiplash, caused by neck hyperextension during a high speed event, often an automotive crash [3]. Anterior longitudinal ligament (ALL) injuries in the cervical spine have been found after whiplash trauma [4]. Examples of spinal ligament injury scenarios in a military setting are repeated loading in military vehicles [5] and aircraft crashes [6]. Because of the high cost and disability associated with these injuries, there is both civilian and military interest in the development of accurate computational models. These computational spine models support investigations designed to prevent or reduce injuries, but need accurate material properties valid for the simulated scenario to provide relevant predictions. Several studies have investigated the biomechanical properties of spinal ligaments at high-rate deformation strains [7-9]. However, few have accounted for their viscoelastic nature; the accurate characterisation of short- and long-term stress relaxation behaviour under loading.

Reference [1] provided a novel viscoelastic model of the anterior longitudinal ligament (ALL), posterior longitudinal ligament (PLL), and ligamentum flavum (LF) using quasilinear viscoelasticity theory (QLV). A range of material constants was found to describe the instantaneous elastic response and relaxation behaviour of the ligaments, including relaxation during loading. This behaviour, however, is derived from the response of many exponential elements spanning the time domain. The majority of relaxation from an exponential term occurs over a single decade on a logarithmic time scale. To capture the ligament response during a longer time frame, multiple exponential terms with a variety of time constants can be summed to form a single relaxation function, often called a Prony Series.

A relaxation function for a material can be derived from the constitutive equation describing its stress-strain behaviour. When the constitutive equation contains integer order differentials (e.g. first derivative, second derivative, etc.), this relaxation function will consist of exponential functions. Principles of fractional calculus

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have been proposed [10] to describe material behaviour, a theoretical basis for their use has been established [11], and fractional derivatives have been shown to naturally appear in the behaviour of real materials [12]. Introducing fractional derivatives in the constitutive equation provides a direct description of materials exhibiting power law relaxation behaviour, spanning multiple decades on a logarithmic time scale using only a few parameters [13,14]. Instead of describing material behaviour at each time scale separately, a general description valid over extended time scales may be possible. Materials exhibiting power law relaxation behaviour are relatively easy to identify, as their pure relaxation behaviour appears as a straight line on a log-log stress relaxation time graph. An illustration of the difference between exponential and power law relaxation behaviour can be seen in Fig. 1. Fractional order viscoelastic models have been used in biological tissues to model stress relaxation of arteries [15], and a fractional viscoelastic analytical model of the periodontal ligament has been proposed [16], but to the knowledge of the authors, no fractional order viscoelastic models based on experimental data of stress relaxation in any human ligaments exist in the scientific literature.

The aim of the current study is to characterise the stress relaxation of ALL cervical spinal ligaments under fast strain rates (up to 80 s^{-1}) using a novel fractional order linear viscoelastic model, with relaxation data obtained from [1]. A comparison is made with an integer order linear viscoelastic model, as used in the original study.

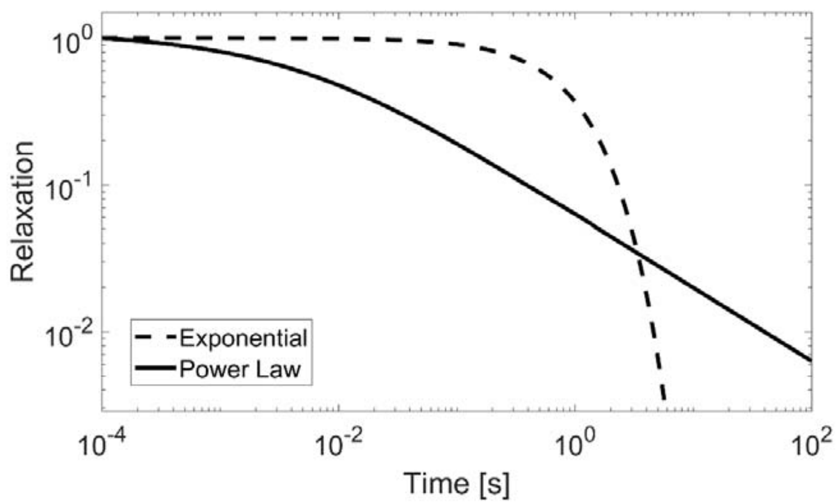

Fig. 1. Relaxation behaviour over extended timescales for an exponential element (dashed line) as used in the integer model, and power law element (solid line) as used in the fractional model in this study. The power law element from the fractional model shows decay in a straight line on the log-log plot, spanning multiple decades.

II. METHODS

All experimental data was obtained from [1]. For an in depth description of materials and methods, one can refer to the original publication. An overview of essential methods is described here.

Cervical Spine Specimens

The cervical spinal ligaments of three male and two female post-mortem human subjects were used in this study, more information on the subjects can be found in Table I. Cervical spines were separated into C3-C4, C5-C6, and C7-T1 functional spinal units (FSUs). Each FSU was further segmented into bone-ligament-bone sections for the anterior longitudinal ligament (ALL), posterior longitudinal ligament (PLL) and ligamentum flavum (LF). In the present manuscript, only the modelling of the ALL is discussed. Soft tissue surrounding the ligaments was maintained for hydration purposes. The bone on both sides of the ligament was potted in aluminium cups using urethane casting resin (Fast Cast, Goldenwest, Cedar Ridge, CA, USA). Specimens were bagged and placed in a water bath [7] to approximate physiological temperature and hydration conditions. After excluding damaged specimens and especially noisy test data, 13 ALL tests were used for modelling.

TABLE I
POST-MORTEM HUMAN SUBJECTS

Specimen ID	Gender	Age (y)	Stature (mm)	Mass (kg)	Spinal level
1	M	60	1780	84	C3-C4
					C5-C6
					C7-T1
2	M	43	1745	107	C3-C4
					C5-C6
					C7-T1
3	F	57	1685	50	C3-C4
					C5-C6
					C7-T1
4	M	66	1680	66	C3-C4
					C5-C6
5	F	61	1600	50	C3-C4
					C5-C6

Mechanical Testing

After mounting the specimens in a universal test machine (Instron, Inc. #8874, Canton, MA, USA), the soft tissue surrounding the ligaments was removed. The mechanical testing was conducted inside an environmental chamber to approximate physiological temperature (37.2 ± 0.6 °C) and humidity (>90%). A detailed schematic of the testing conditions can be found in found in [7], companion paper to [1]. Engineering strain (ε_E) was defined as the ratio of displacement (Δl), to initial length (l_0). Initial length was measured by digital calipers as the distance between the cranial endplate of the superior vertebral body and the caudal endplate of the inferior vertebral body after applying a 4 N tension preload to the specimen. The average $\pm$ standard deviation of l_0 for the used specimens was 3.67 $\pm$ 0.75 mm. Preconditioning for each ligament was done using a 10% ε_E sinusoidal input at 2 Hz for 120 cycles. Following preconditioning, tensile ramp holds at 25% ε_E (R_{25}) and 50% ε_E (R_{50}) tests were performed with a 10 minute recovery time between each test. The duration of each ramp was approximately 10 ms, limited by the displacement rate of the universal test machine, and the tension strain was held for 1 min before returning the specimen to the neutral position. Strain rates reached up to 50 s^{-1} in the R_{25} tests and up to 80 s^{-1} in the R_{50} tests. Axial force (F) and displacement (Δl) were sampled with a rate of 10,000 Hz. The analysis in the current manuscript uses only the first 1,100 ms of this recording.

Stress and Strain Conversion

Measured force and displacement were converted to true stress (σ_T) and true strain (ε_T) as described in [7]. Original ligament cross-sectional area (A_0) was assumed proportional to the width of the cranial endplate of the superior vertebral body (VBW) and A_0 was determined by scaling as

$$A_0 = A_{50}\frac{VBW}{VBW_{50}} \tag{1}$$

where A_{50} and VBW_{50} are ligament cross-sectional areas and vertebral body widths respectively for the 50$^{\mathrm{th}}$ percentile male specimens from stud. The ligaments were assumed incompressible during testing, so

$$A\,l = A_0 l_0 \tag{2}$$

can be used to relate instantaneous cross-sectional area and length A and l to original cross-sectional area and length A_0 and l_0. True stress was calculated as

$$\sigma_T = \frac{F}{A} = \frac{F}{A_0}\cdot\frac{l}{l_0} = \sigma_E(1+\varepsilon_E) \tag{3}$$

where σ_E is the engineering stress, defined as the ratio of force F to original cross-sectional area A_0. True strain then was defined as

$$\varepsilon_T = \int d\varepsilon = \int_{l_0}^{l} \frac{dl}{l} = \ln\frac{l}{l_0} = \ln\frac{l_0 + \Delta l}{l_0} = \ln(1 + \varepsilon_E) \tag{4}$$

where ε_E is the engineering strain as mentioned earlier.

Viscoelastic Models

Reference [1] found that the instantaneous elastic behaviour of the ligaments can be very closely approximated by a linear model. Models analysed in this study assume a linear relationship between the instantaneous elastic stress and engineering strain. True ligament tensile stress σ_T was modelled using a hereditary integral [17]:

$$\sigma_T(t) = \int_0^t G_{red}(t - t')\, K \frac{d\varepsilon_T}{dt'} dt' \tag{5}$$

where G_{red} is the reduced relaxation function, K is the instantaneous elastic parameter, t is time and t' is a dummy variable used for integration. A linear instantaneous elastic relationship is used here: $\sigma_{el} = K\varepsilon_T$. As in the original paper [1], an exponential formulation for the instantaneous elastic relationship was investigated and found not to improve overall fit results, for all models used.

Two different forms of G_{red} are compared, an integer order model and a fractional order model. The integer order reduced relaxation function is expressed as a sum of exponential terms [1]:

$$G_{red}(t) = G_\infty + \sum_{i=1}^{n} G_i e^{-\frac{t}{\tau_i}} \quad with \quad G_\infty + \sum_{i=1}^{n} G_i = 1 \tag{6}$$

with G_∞ the steady-state relaxation coefficient, and n the number of G_i relaxation coefficients with time constants τ_i. The values of the τ_i were set to decade values, i.e., 1 s, 100 ms, 10 ms, 1 ms. An example of the relaxation behaviour of one of these exponential terms can be seen in Fig. 1 (dashed line).Models containing up to five relaxation coefficients (n = 4) were evaluated and compared to results from the fractional order model.

The reduced relaxation function for the fractional order model was expressed as:

$$G_{red}(t) = \sum_{i=1}^{m} \left(E_i + E_i((\beta_i \rho_i)^{\alpha_i} - 1)\, E_{\alpha_i}(-(\beta_i t)^{\alpha_i})\right) \tag{7}$$

where m is the number of fractional elements in parallel, with each element described by α_i, the fractional order exponent, β_i the relaxation constant, ρ_i the retardation constant and E_i, the elastic modulus. $E_\alpha(z)$ is the Mittag-Leffler function, defined by the infinite series [18]:

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)} \tag{8}$$

For α = 1, the Mittag-Leffler function becomes an exponential function and the fractional model is equivalent to the integer model. Models containing up to two fractional order elements (m = 2) were evaluated. An example of the relaxation behaviour a fractional order element can be seen in Fig. 1 (solid line).

The hereditary integral in Equation (5) was numerically integrated. Optimisation of instantaneous elastic parameter K, integer model parameters G_∞, G_i, τ_i, and fractional model parameters $E_i, \alpha_i, \beta_i, \rho_i$, to fit model stress to measured stress was done with least squares fitting and a trust region optimisation algorithm in MATLAB® (MathWorks, Inc., Natick, MA, USA). Parameters were restricted to strictly positive values. After comparing parameter values for the different ligaments, fixed values of fractional exponents α_i were chosen. Parameters were optimised for R_{50} tests and then used to generate a prediction fit for the R_{25} tests to validate the model.

Models were compared by visual inspection, the R^2 value, and sum of squared error (SSE) ratios. The reason the ratio of SSE is used rather than an absolute value is that size differences and oscillations in the stress data produce model fits with SSE values spanning multiple orders of magnitude. A comparison between models for one ligament is possible, but SSE is not the right metric for a comparison between different ligaments.

III. RESULTS

Results were obtained for a nine-parameter integer order model (n = 4), and an eight-parameter fractional order model (m = 2) for the R_{50} tests and compared to each other. After observing initial optimisation results, α_1 and α_2 values were set fixed to 0.1 and 0.8 in the fractional order model. Both models provided excellent results for modelling ligament behaviour in R_{50} tests with R^2 values average $\pm$ SD being 0.989 $\pm$ 0.010 for the integer order model and 0.989 $\pm$ 0.009 for the fractional order model. The median ratio of integer model SSE to fractional model SSE was 1.006, with the fractional model providing a slightly better result. The average SSE ratio however was 0.965, showing a better average fit for the integer order model. The optimisation results can be found in Table II and Table III.

A seven-parameter integer order model (n = 3) was constructed by eliminating from the nine-parameter model the relaxation coefficient and time constant with the smallest contribution. Contrary to the findings in [1], the G_4, τ_4= 1 ms term was found to have the smallest contribution to the model fit, with its contribution becoming negligibly small (< 10^{-7}) in 8 of the 13 cases. All four possible seven-parameter models were analysed, and it was confirmed that the G_1, G_2, G_3 model provided the best results.

With an R^2 value of 0.988 $\pm$ 0.009, the seven-parameter model performed almost as well as the nine-parameter model and the eight-parameter fractional model. The ratio of seven-parameter integer SSE to eight-parameter fractional SSE had a median of 1.010 and an average of 1.17. The results are represented in Table IV. A representative stress-time history with model fits for the three described models can be found in Fig. 2.

TABLE II
8 PARAMETER
FRACTIONAL ORDER MODEL

Parameter	Mean $\pm$ SD
α_1	0.1
α_2	0.8
β_1	7.54 $\pm$ 16.96
β_2	71.65 $\pm$ 43.63
ρ_1	4.21 $\pm$ 4.18
ρ_2	1.77 $\pm$ 3.00
E_1	0.98 $\pm$ 0.01
E_2	0.02 $\pm$ 0.01
K	336.1 $\pm$ 934.0
R^2	0.989 $\pm$ 0.009
SSE	1 (reference)

TABLE III
9 PARAMETER
INTEGER ORDER MODEL

Parameter	Mean $\pm$ SD
τ_1	1
τ_2	0.1
τ_3	0.01
τ_4	0.001
G_1	0.113 $\pm$ 0.026
G_2	0.082 $\pm$ 0.030
G_3	0.213 $\pm$ 0.081
G_4	0.088 $\pm$ 0.157
G_∞	0.503 $\pm$ 0.141
K	863 $\pm$ 2502
R^2	0.989 $\pm$ 0.010
SSE	0.965 $\pm$ 0.112

TABLE IV
7 PARAMETER
INTEGER ORDER MODEL

Parameter	Mean $\pm$ SD
τ_1	1
τ_2	0.1
τ_3	0.01
G_1	0.128 $\pm$ 0.031
G_2	0.083 $\pm$ 0.034
G_3	0.253 $\pm$ 0.092
G_∞	0.536 $\pm$ 0.116
K	836 $\pm$ 2506
R^2	0.988 $\pm$ 0.009
SSE	1.169 $\pm$ 0.569

The optimised parameter results of the R_{50} tests for all three models were used to predict the stress in the R_{25} tests as a validation of the models. The average $\pm$ SD R^2 values found were 0.953 $\pm$ 0.035 for the fractional order model, 0.953 $\pm$ 0.035 for the nine-parameter integer order model, and 0.954 $\pm$ 0.032 of the seven-parameter integer order model. The ratios of SSE between the integer order models and the fractional order model had an average of 1.001 for both the nine-parameter and the seven-parameter integer order model.

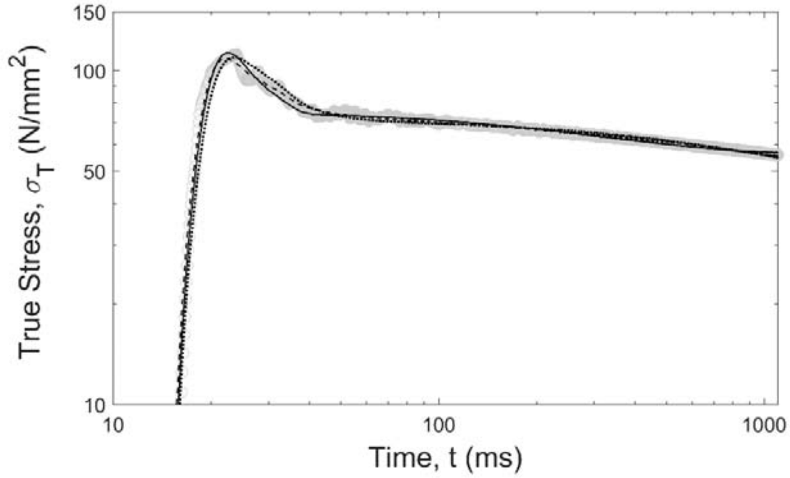

Fig. 2. Representative model fit to experimental stress-time response (grey circles) for an R_{50} test (50% engineering strain). The solid line is the fractional eight-parameter model fit, the dotted line is the integer seven-parameter model fit, and the dashed line is the integer nine-parameter model fit. Both time and stress axis have a logarithmic scale. Straight line decay on this log-log plot suggest power law relaxation behaviour in the ligament.

IV. Discussion

A fractional viscoelastic model was developed and applied to model stress relaxation of the anterior longitudinal ligament (ALL) in the cervical spine under high-rate loading conditions. This fractional model is a new approach for describing experimental stress relaxation behaviour in ligaments.

The results of the R_{50} model optimisation suggest that the fractional order model does not outperform the traditional integer order models in this scenario, providing similar results for a similar number of parameters used. Considering the added mathematical complexity and computations necessary to operate the fractional model compared to the traditional model with exponential functions, the traditional integer model is the preferred choice for modelling the ALL.

Upon close examination of the ligament relaxation behaviour, some of the ligaments deviated from a straight line in a log-log plot, this is a possible reason why the fractional model provided better results for some ligaments but not for others. The methodology used in this study can be applied to different materials, biological or otherwise, when power law relaxation is suspected.

Contrary to integer order models, fractional models have the potential to be valid over extended time scales because they are theoretically exact for power law relaxation behaviour, increasing the value for computational problems with both short term acute and long term physiological response. Fractional models do not offer a *one size fits all* solution, because they are computationally more expensive than traditional models, so the inherent relaxation behaviour of a material should be evaluated before a computational model is chosen. Straight line relaxation behaviour on a log-log plot suggests power law relaxation, which could be modelled using fractional calculus.

For the prediction of the R_{25} stresses based on the parameters optimised for behaviour of the ligaments in the R_{50} test, all three models performed similarly. This is because the difference in ligament behaviour between the R_{25} test and the R_{50} test is larger than the difference between the model stresses and the R_{50} stress. Any distinction between the models is small in comparison. Surprisingly, the seven-parameter integer order model had an R^2 value slightly higher than the other two models, but all three models provided a good fit.

The stress strain histories used in this analysis only contained relaxation data up to 1.1 s after the ramp tensile load, while in the original study relaxation behaviour up to 1 minute after the ramp was recorded. Future studies should include both short and long time scales to investigate where a fractional model might perform better.

V. CONCLUSIONS

Fractional order viscoelastic models provide similar results to traditional integer order models for relaxation behaviour in cervical spinal anterior longitudinal ligaments. Due to the added complexity of the fractional model, the traditional model is preferred in this case. Fractional order models might be applicable for other materials or loading scenarios, because they have the potential to be valid over extended time scales. Inherent power law relaxation shows straight line decay on a log-log plot of stress over time. Fractional calculus is theoretically exact for power law behaviour. Future studies containing both very short and long time scales are recommended to distinguish between fractional and traditional models.

VI. ACKNOWLEDGEMENT

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